Kinetic Electron Closures for Electromagnetic Simulation of Drift and Shear-Alfvén Waves

Bruce I. Cohen, Andris M. Dimits, and W. M. Nevins
University of California Lawrence Livermore National Laboratory,
Livermore, California 94550

Yang Chen and Scott E. Parker
Department of Physics, University of Colorado at Boulder
Boulder, Colorado 80309

* This work was supported by the U.S. Department of Energy under Contract No. W-7405-ENG-48 at University of California Lawrence Livermore National Laboratory. This work is part of the SCIDAC Plasma Microturbulence Project sponsored by DOE OFES.

Kinetic Electron Closures for Electromagnetic Simulation -- Outline

1. A practical algorithm for particle simulation of electromagnetic drift-wave turbulence and transport with kinetic ions and electrons
2. Examples – Simulations of kinetic shear Alfvén waves, and collisionless drift wave and ion-temperature-gradient instabilities at finite β in a two-dimensional unsheared slab
3. Extension to toroidal flux-tube algorithm
4. Summary --

   Successful particle simulations of shear-Alfvén waves and electromagnetic drift-wave and ITG instabilities with kinetic electrons for $\beta m_i / m_e > 1$ (hot core plasmas) in slab. Toroidal code being debugged.

Related work within the SciDAC Plasma Microturbulence Project: continuum methods -- GS2 by W. Dorland, et al., and GYRO by Waltz and Candy; particle methods -- Z. Lin and L. Chen, W. Lee.

How to Accommodate $\beta m_i/m_e$>>1 ? -- Hybrid II Electromagnetic Algorithm

- Extend the “massless” electron hybrid model of Parker, *et al.* and P. Snyder (Ph.D. thesis, Princeton U., 1999) to include drift-kinetic electrons.

- Consider the modified electron momentum equation (Ohm’s law) in slab geometry:

$$en_{0e}\vec{E}\ \ \hat{b}^{(0)} = -\ \ {}_{\|}P_{\|e} + \frac{\delta\vec{B}}{B}\qquad en_{0e}\phi - n_{0e}m_e(\partial/\partial t + \vec{v}_{ExB}\qquad)u_{\|e}$$

where $\ {}_{\|}P_{\|e} = \ {}_{\|}P_{\|e}^{(0)} + T_{\|e}^{(0)}\ \ {}_{\|}\delta n_e^{(0)} + n_{0e}\ \ {}_{\|}\delta T_{\|e}$ with $\ {}_{\|}(T_{\|e}^{eq} + \delta T_{\|e}) = 0$, $T_{\|e}^{(0)}$ is a constant, $\delta n_e^{(0)} = \delta n_e - \Delta n_e^K$ =electron fluid density, $\Delta n_e^K = \int d^3 v h_e$ is the *split-weight* δf kinetic increment, and δn_e = total perturbed density consistent with moment of *split-weight* electron distribution function (like Lin and Chen, 2001):

$$f_e = f_M(\vec{x},\vec{v}) + \left({\delta n_e^{(0)}}\big/{n_{0e}}\right) f_M(\vec{v}) + h_e(\vec{x},\vec{v})$$

- Use Ohm’s law to advance $A_{\|}$, ${\partial A_{\|}}\big/{c\partial t} = (\vec{E} + \quad \phi)\ \ \hat{b}^{(0)} = \ldots$

Hybrid II Electromagnetic Algorithm (cont'd)

- With updated $A_{\|}$ use Ampere's law to determine parallel electron current:

  $\Gamma_{\|e} = n_{0e} u_{\|e} = \frac{c^2}{4\pi e} \quad {}^2 \frac{A_{\|}}{c} + \bar{\Gamma}_{\|i}$, where $\bar{\Gamma}_{\|i}$ is the gyrokinetic parallel ion current.

- Use the electron continuity equation to advance the total electron density:

$$\frac{\partial \delta n_e}{\partial t} + n_{0e}(\vec{B}^{(0)} + \delta\vec{B} \quad) \qquad \frac{u_{\|e}}{B} + \vec{v}_{E\times B} \qquad (n_e^{eq} + \delta n_e) = 0$$

(assumes no magnetic curvature)

- Determine the electric potential ϕ from the quasineutrality relation using the updated electron and gyrokinetic ion densities

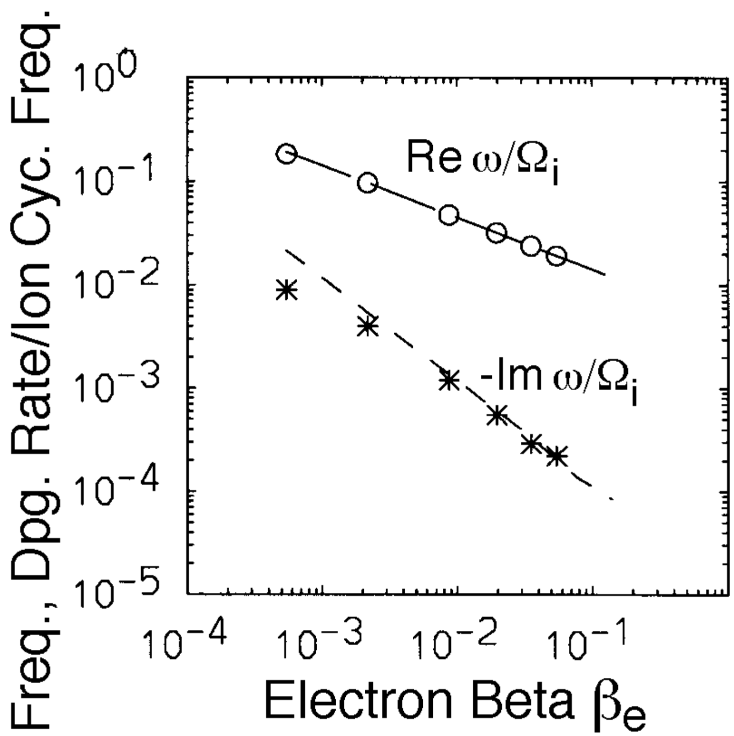
- Advance the gyrokinetic ions and the drift-kinetic electrons with same Δt.

- From drift-kinetic equation for electrons with split-weights (after cancellations),

$$\frac{dw_j^e}{dt} = (\vec{\mathrm{v}}_{ExB} \quad \hat{x} + \mathrm{v}_{\|} \frac{\delta B_x}{B_0})\kappa_{Te}(\frac{\mathrm{v}^2}{\mathrm{v}_e^2} - \frac{3}{2}) + \quad {}_{\|} u_{\|e}$$

  using $|\Delta n_e^K / \delta n_e^{(0)}| << 1$ as an expansion parameter.

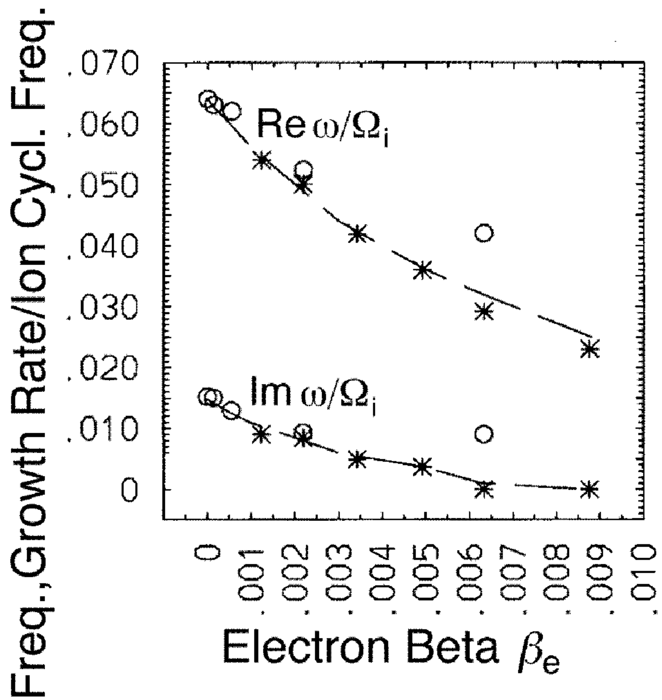
- Simulations of kinetic shear-Alfvén waves in slab.
  Parameters: $k_y\rho_s = \pi/8$, $T_e=T_i$, $B_y/B_0=0.01$, $\rho_s=2\Delta y$, 32x32 grid, (0,1) mode
  theory - - -   simulation results: o = $\mathrm{Re}\,\omega/\Omega_i$, $= -\mathrm{Im}\,\omega/\Omega_i$ Landau dpg. rate

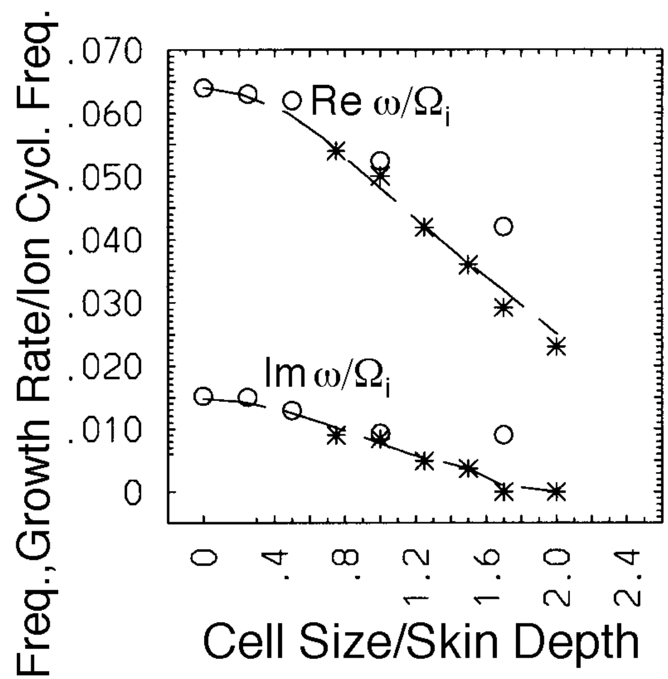


- No restriction on $\omega_{pe}\Delta y/c$ and results are similar to Z. Lin and L. Chen's 2001 reported results. (As $\beta_e m_i / m_e \to 0$ the algorithm fails and goes unstable.)

Extended Hybrid II δf Algorithm – Collisionless Drift-Wave Slab Simulations

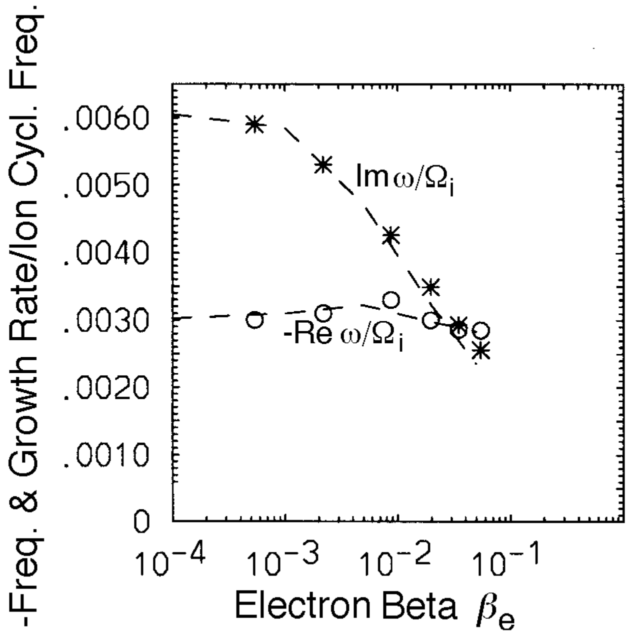
- δf slab simulations of collisionless drift-wave instability with no magnetic shear. Parameters: $k_y\rho_s = \pi/4$, $\rho_s/L_n = 0.2$, $T_e = T_i$, $B_y/B_0 = 0.01$, $\rho_s = 2\Delta y$, 16x16 grid, and (0,1) mode, theory (J. Cummings Ph.D. thesis) - - -,
  o= standard δf simulation, = extended Hybrid II code with kinetic electrons.





- The Hybrid II algorithm gives good results for $\beta m_i/m_e > 1$ and any skin depth, while the standard δf simulation fails except for $\beta m_i/m_e \leq 1$ and $\Delta y < c/\omega_{pe}$.

- 2D Hybrid II simulations of shearless ITG accurate for $\beta m_i/m_e > 1$ and no constraint on the skin depth, i.e., c/ω_{pe} relative to the cell size Δx. Accommodates finite η_e.



- 2D slab simulations with no shear, θ=0.01, T_e=T_i, η_i =η_e=4, ρ_s/L_n=0.1, Ω_e/ω_{pe}=1, m_i/m_e=1836, ρ_s=2Δx, 32×32 grid. Frequency and growth rates for the (0,1) mode ($k_y \rho_s \approx \pi/8$) vs. $\beta_e = (\omega_{pe}\Delta x/c)^2(\rho_s/\Delta x)^2(m_e/m_i)(\omega_{pe}/\Omega_e)^2$ theory (- - -) (J. Cummings, Ph.D. Thesis, 1995) (±10% error bars in obs. Re ω)

- Consider the consequences of the explicit backward differencing of the inertia term in the Ohm's law, which leads to a numerical instability at low β:

$$en_{0e}\vec{E}\ \hat{b}^{(0)} = -\ {}_{\|}P_{\|e} + \frac{\delta\vec{B}}{B}\quad en_{0e}\phi - n_{0e}m_e(\partial/\partial t + \vec{v}_{ExB}\quad)u_{\|e},\ldots\partial u_{\|e}/\partial t = \Delta t^{-1}(u^n_{\|e} - u^{n-1}_{\|e})$$

- A heuristic stability analysis yields the following dispersion relation for shear-Alfvén waves:

$$(\lambda^{-1} + \omega^2_{pe}/k^2c^2)(\lambda-1)^2/4\lambda = \omega^2_h\Delta t^2/4,\ldots\lambda \equiv \exp(-i\omega\Delta t)$$

- Limits---
  Electrostatic: $\omega^2_{pe}/k^2c^2 = (\beta_e m_i/m_e)/k^2\rho_s^2 << 1 \quad \rightarrow \lambda = 1/(1\pm\omega_h\Delta t)$

  $|\lambda|>1$ ***instability*** !!

  Electromagnetic: $\omega^2_{pe}/k^2c^2 = (\beta_e m_i/m_e)/k^2\rho_s^2 >> 1$ and $k_{\|}v_A\Delta t/2 < 1$

  stability with $\omega\Delta t/2 = \pm\frac{1}{2}k_{\|}v_A\Delta t(1-\varepsilon/2) - i\varepsilon(\frac{1}{2}k_{\|}v_A\Delta t)^2, \varepsilon \equiv k^2c^2/\omega^2_{pe} << 1$

  General dispersion relation is a cubic (for a cold plasma), and there is numerical ***stability*** for $\omega^2_{pe}/k^2c^2 = (\beta_e m_i/m_e)/k^2\rho_s^2 > O(1)$.

Hybrid II Code – Implicit Solution of $u_{\|e}$ to Improve Numerical Stability

- If we Introduce implicit time differencing of $u_{\|e}$ in the electron inertia term in Ohm’s law, then

$$en_{0e}\vec{E}\ \ \hat{b}^{(0)} = -\ \ _{\|}P_{\|e} + \frac{\delta\vec{B}}{B}\quad en_{0e}\phi - n_{0e}m_e(\partial/\partial t + \vec{v}_{ExB}\quad)u_{\|e},\ldots\partial u_{\|e}/\partial t = \Delta t^{-1}(u_{\|e}^{n+1} - u_{\|e}^{n})$$

- Next solve for $A_{\|}^{n+1}$ in terms of $u_{\|e}^{n+1}$ and substitute into Ampere’s law to obtain an elliptic equation for $u_{\|e}^{n+1}$

$$(-\ \ ^2 + \omega_{pe}^2/c^2)u_{\|e}^{n+1} = -\ \ ^2(\ldots)^n$$

- We would expect this modification of the hybrid algorithm to have improved numerical stability for smaller values of $\omega_{pe}^2/k^2c^2 = (\beta_e m_i/m_e)/k^2\rho_s^2$. How much smaller?

- However, the hybrid algorithm will remain ***poorly posed*** as $\beta \to 0$ because of the presence of $A_{\|}$ and use of Ampere’s law which becomes 0=0.

- Determine the parallel electric field from the modified electron momentum equation (Ohm's law) including toroidicity (ref: P. Snyder and G. Hammett)

$$en_{0e}\vec{E}\ \ \hat{b}^{(0)} = -\ \ {}_{\|}P_{\|e} + \frac{\delta\vec{B}}{B}\quad en_{0e}\phi - n_{0e}m_e(\partial/\partial t + \vec{v}_{ExB}\quad)u_{\|e}$$

$$-\left(\tfrac{1}{2}\delta P_{\ e} - \delta P_{\|e}\right)\hat{b}^{(0)}\quad \ln B$$

where $\ {}_{\|}P_{\|e} =\ \ {}_{\|}P^{(0)}_{\|e} + T^{(0)}_{\|e}\ \ {}_{\|}(\delta n_e - \delta n_e^{K}) + n_{0e}\ \ {}_{\|}\delta T_{\|e}$ with $\ {}_{\|}(T^{eq}_{\|e} + \delta T_{\|e}) = 0$.

- Use Ohm's law to advance $A_{\|}$, $\partial A_{\|} / c\partial t = (\vec{E} + \quad \phi)\ \hat{b}^{(0)} = \ldots$
- With the updated $A_{\|}$ use Ampere's law to determine parallel electron flux:

$\Gamma_{\|e} = n_{0e}u_{\|e} = \frac{c^2}{4\pi e}\quad {}^{2}\ \frac{A_{\|}}{c} + \bar{\Gamma}_{\|i}$, where $\bar{\Gamma}_{\|i}$ is the gyrokinetic parallel ion current.

- Use the electron continuity equation to advance the total electron density:

$$\frac{\partial\delta n_e}{\partial t} + n_{0e}(\vec{B}^{(0)} + \delta\vec{B}\ \ )\quad \frac{u_{\|e}}{B} + \vec{v}_{E\times B}\quad n_e$$

$$+\frac{1}{m_e\Omega_e B^2}(\vec{B}\times\quad B)\quad (\tfrac{1}{2}\delta P_{\ e} + \delta P_{\|e}) + \frac{2n_{0e}}{B^3}(\vec{B}\times\quad B)\quad \phi = 0$$

Hybrid II Toroidal Electromagnetic Algorithm (cont'd)

- Determine the electric potential ϕ from the quasineutrality relation using the updated electron and gyrokinetic ion densities:

$$^2\phi - \frac{\tau(\phi - \tilde{\phi})}{\lambda_D^2} = 4\pi e\left(\delta\bar{n}_i - \delta n_e\right)$$

- Advance the gyrokinetic ions and the drift-kinetic electrons including the toroidal drifts: $\vec{v}_{gs} = v_{||}\hat{b} + \vec{v}_{E\times B} + \vec{v}_{ds}$, $\vec{v}_{ds} = \frac{v_{||}^2 + v^2/2}{\Omega_s B^2}\vec{B}\times \;\; B$, $\Omega_s \equiv q_s B_0/m_s c$

  and mirroring.

$$\dot{v}_{||} = (q_s/m_s)\hat{b} \;\; \vec{E} - (\mu_s/m_s)\hat{b} \;\; B + v_{||}(\hat{b} \;\; \hat{b}) \;\; \vec{v}_{E\times B}$$

- From drift-kinetic equation for electrons with split-weights (after cancellations)

$$\frac{d}{dt}w_i^e = (\kappa_e - \kappa_{ne})\hat{x} \;\; (\vec{v}_{E\times B} + v_{||}\hat{b}) - \vec{v}_{de} \;\; \delta n_e/n_{0e} + \vec{B} \;\; (u_{||e}/B) + (v_{||}/v_e^2)(\tfrac{\partial}{\partial t} + \vec{v}_{E\times B} \;\;)u_{||e}$$
$$+ v_{||}(\hat{b}^{(0)} \;\; \ln B)(\tfrac{1}{2}\delta p_{\;e} - \delta p_{||e})/(n_{0e}T_e^{(0)}) + \vec{v}_{E\times B} \;\; (\varepsilon_{||}\hat{b} \;\; \hat{b} + \tfrac{1}{2}\varepsilon \;\; \ln B)/T_e^{(0)}$$
$$+ (n_{0e}m_e\Omega_e B^2)(\vec{B}\times \;\; B) \;\; (\delta p_{||e} + \tfrac{1}{2}\delta p_{\;e}) + (2c/B^3)(\vec{B}\times \;\; B) \;\; \phi$$

- Use flux-tube coordinates: $x=r-r_0$, $y=(r_0/q_0)(q\theta-\zeta)$, $z=q_0R_0\theta$

Initial Comparison Run of Toroidal Hybrid II and GEM Codes

- Toroidal ITG simulations with 32x32x32 grid, $\beta=5\text{x}10^{-4}$, $\eta_i=3.5$, $q_0=1.4$, kinetic electrons and ions, $m_i/m_e=1837$, comparing GEM (conventional low-β code) to toroidal Hybrid II preliminary results. Toroidal Hybrid II is being debugged.